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Nonlinear triple-point problems with change of sign

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Abstract

In this paper, we study the existence of at least one or two positive solutions to the second-order triple-point nonlinear boundary value problem

$$y''(x) + h(x)f(y(x)) = 0, \quad x \in [a, b]$$
$$y(a) = \alpha y(\eta), \qquad y(b) = \beta y(\eta),$$

where $0 < \alpha < \beta < 1$ and $\eta \in (a, b)$. Here h changes sign in η. As an application, we also give some examples to demonstrate our results.

Keywords: Positive solutions; Fixed-point theorems; Cone; Alternating coefficient

1. Introduction

Three-point boundary value problems for differential equations have been studied in recent years. In most of these studies, the function h is assumed to be nonnegative or nonpositive (see [1–5]). Liu [6] has studied the existence of positive solutions of the second-order boundary value problem

$$\begin{cases} y''(x) + \lambda a(x) f(y(x)) = 0, \quad 0 < x < 1, \\ y(0) = 0, \qquad y(1) = \beta y(\eta), \end{cases} \tag{1.1$'$}$$

where λ is a positive parameter, $0 < \beta < 1, 0 < \eta < 1$, the function a is an alternating coefficient on $[0, 1]$. He used the Krasnoselskii fixed-point theorem and obtained some simple criteria for the existence of at least one positive solution of the BVP (1.1′).

In this paper, we shall use Krasnoselskii fixed-point theorem and Avery–Henderson fixed-point theorem to investigate the existence of at least one positive solution and of at least two positive solutions respectively to triple-point boundary value problem

$$\begin{cases} y''(x) + h(x) f(y(x)) = 0, \quad x \in [a, b], \\ y(a) = \alpha y(\eta), \qquad y(b) = \beta y(\eta), \end{cases} \tag{1.1}$$

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where $0 < \alpha < \beta < 1$, $a < \eta < b$, h changes sign in η.

We will assume that the following conditions are satisfied.

(H1) $f : [0, +\infty) \to (0, +\infty)$ is continuous and nondecreasing.

(H2) $h : [a, b] \to \mathbb{R}$ is continuous and such that $h(x) \geq 0$, $x \in [a, \eta]$; $h(x) \leq 0$, $x \in [\eta, b]$. Moreover, it does not vanish identically on any subinterval of $[a, b]$.

(H3) There exists a constant $\tau \in (a + \frac{\beta-\alpha}{1-\alpha}(\eta - a), \eta)$ such that, for all $x \in [0, b - \eta]$ the function

$$H(x) = \delta h^{+}(\eta - \delta x) - \frac{1}{\Lambda} h^{-}(\eta + x) \geq 0,$$

where $h^{+}(x) = \max\{h(x), 0\}$, $h^{-}(x) = -\min\{h(x), 0\}$, and

$$\delta = \frac{\eta - \tau}{b - \eta}, \qquad \Lambda = \frac{\beta - \alpha}{b - a} \min\left\{\frac{\beta}{1 - \alpha}(\eta - a), b - \eta, \frac{\alpha}{1 - \alpha}(b - a)\right\}.$$

Our (H3) condition is a generalization of the condition (H4) of Liu [6].

2. Preliminary lemmas

In this section, we present auxiliary lemmas which will be used later.

First, define the number D by

$$D = \alpha(\eta - b) + \beta(a - \eta) + b - a.$$

Lemma 2.1. *Let $D \neq 0$. Then for $k \in \mathcal{C}[a, b]$, the problem*

$$\begin{cases} y''(x) + k(x) = 0, & x \in [a, b], \\ y(a) = \alpha y(\eta), & y(b) = \beta y(\eta) \end{cases} \tag{2.1}$$

has the unique solution

$$y(x) = -\int_a^x (x - s)k(s)\mathrm{d}s + \frac{\alpha(x - b) + \beta(a - x)}{D} \int_a^{\eta} (\eta - s)k(s)\mathrm{d}s + \frac{\alpha(\eta - x) + x - a}{D} \int_a^b (b - s)k(s)\mathrm{d}s.$$

Let $G(x, s)$ be the Green's function for the problem (2.1). A direct calculation gives the following:

$$G(x, s) = \begin{cases} G_1(x, s), & a \leq x \leq \eta, \\ G_2(x, s), & \eta < x \leq b, \end{cases}$$

where

$$G_1(x, s) = \begin{cases} g_{11}(x, s) = \dfrac{[\beta(x - \eta) + b - x](s - a)}{\alpha(\eta - b) + \beta(a - \eta) + b - a}, & a \leq s \leq x, \\ g_{12}(x, s) = \dfrac{\alpha(b - \eta)(s - x) + [\beta(s - \eta) + b - s](x - a)}{\alpha(\eta - b) + \beta(a - \eta) + b - a}, & x < s \leq \eta, \\ g_{13}(x, s) = \dfrac{[\alpha(\eta - x) + x - a](b - s)}{\alpha(\eta - b) + \beta(a - \eta) + b - a}, & \eta < s \leq b, \end{cases}$$

and

$$G_2(x, s) = \begin{cases} g_{21}(x, s) = \dfrac{[\beta(x - \eta) + b - x](s - a)}{\alpha(\eta - b) + \beta(a - \eta) + b - a}, & a \leq s \leq \eta, \\ g_{22}(x, s) = \dfrac{(b - x)[\alpha(\eta - s) + s - a] + \beta(x - s)(\eta - a)}{\alpha(\eta - b) + \beta(a - \eta) + b - a}, & \eta < s \leq x, \\ g_{23}(x, s) = \dfrac{[\alpha(\eta - x) + x - a](b - s)}{\alpha(\eta - b) + \beta(a - \eta) + b - a}, & x < s \leq b. \end{cases}$$

Remark 2.1. $G(x, s) \geq 0$ for $(x, s) \in [a, b] \times [a, b]$.

Now, consider the Banach space of continuous functions on $[a, b]$ with the norm

$$\|y\| = \max\{|y(x)| : x \in [a, b]\}.$$

Set

$$\mathcal{C}_0^+[a, b] = \{y \in \mathcal{C}[a, b] : \min_{x\in[a,b]} y(x) \geq 0 \text{ and } y(a) = \alpha y(\eta), y(b) = \beta y(\eta)\}.$$

Denote a cone $\mathcal{P}$ in $\mathcal{C}[a, b]$ given by

$$\mathcal{P} = \{y \in \mathcal{C}_0^+[a, b] : y(x) \text{ is concave on } [a, \eta], \text{ and convex on } [\eta, b]\}.$$

From the definition of Green's function G, it is clear that the solutions of the boundary value problem (1.1) are the fixed points of the operator

$$Ay(x) = \int_a^b G(x, s)h(s) f(s, y(s))\mathrm{d}s, \quad x \in [a, b].$$

The following two lemmas give the inequalities concerning the solution of BVP (1.1).

Lemma 2.2. *Let $y \in \mathcal{P}$ and*

$$\gamma(x) = \begin{cases} \dfrac{(1-\alpha)x + \alpha\eta - a}{\eta - a}, & x \in [a, \eta], \\ \dfrac{(\beta - 1)x + b - \beta\eta}{b - \eta}, & x \in [\eta, b]. \end{cases}$$

Then

$$y(x) \geq \gamma(x)y(\eta), \quad x \in [a, \eta] \quad \textit{and} \quad y(x) \leq \gamma(x)y(\eta), \quad x \in [\eta, b].$$

Proof. Since $y \in \mathcal{P}$, then y is concave on $[a, \eta]$, convex on $[\eta, b]$, $y(a) = \alpha y(\eta)$ and $y(b) = \beta y(\eta)$. Hence, for $x \in [a, \eta]$, we have

$$y(x) \geq y(a) + \frac{y(\eta) - y(a)}{\eta - a}(x - a) = \frac{(1-\alpha)x + \alpha\eta - a}{\eta - a} y(\eta),$$

for $x \in [\eta, b]$, we have

$$y(x) \leq y(b) + \frac{y(\eta) - y(b)}{\eta - b}(x - b) = \frac{(\beta - 1)x + b - \beta\eta}{b - \eta} y(\eta).$$

Hence,

$$y(x) \geq \gamma(x)y(\eta), \quad x \in [a, \eta], \quad \text{and} \quad y(x) \leq \gamma(x)y(\eta), \quad x \in [\eta, b]. \quad \square$$

Lemma 2.3. *Let $y \in \mathcal{P}$ and*

$$\mu = \min\left\{\frac{\beta - \alpha}{1 - \alpha}, \frac{\eta - \tau}{\eta - a}\right\}. \tag{2.2}$$

Then

$$y(x) \geq \mu\|y\|$$

for $x \in [a + \frac{\beta-\alpha}{1-\alpha}(\eta - a), \tau]$.

Proof. Let $y \in \mathcal{P}$, then y is concave on $[a, \eta]$, convex on $[\eta, b]$. Since $y(b) < y(\eta)$, then

$$\|y\| = \max_{x\in[a,b]} |y(x)| = \max_{x\in[a,\eta]} |y(x)|.$$

Set

$$\omega = \inf\{\xi \in [a, \eta] : \max_{x\in[a,\eta]} y(x) = y(\xi)\}.$$

Case (i). $x \in [a, \omega]$. Since y is concave on $[a, \eta]$, we have

$$\begin{aligned} y(x) &\geq y(a) + \frac{y(\omega) - y(a)}{\omega - a}(x - a) \\ &= \frac{x-a}{\omega-a} y(\omega) + \frac{\omega - x}{\omega - a} y(a) \\ &\geq \frac{x-a}{\eta-a} y(\omega) = \frac{x-a}{\eta-a}\|y\|. \end{aligned}$$

Case (ii). $x \in [\omega, \eta]$. Since y is convex on $[\eta, b]$, we have

$$\begin{aligned} y(x) &\geq y(\omega) + \frac{y(\eta) - y(\omega)}{\eta - \omega}(x - \omega) \\ &= \frac{x-\omega}{\eta-\omega} y(\eta) + \frac{\eta - x}{\eta - \omega} y(\omega) \\ &\geq \frac{\eta-x}{\eta-a} y(\omega) = \left(1 - \frac{x-a}{\eta-a}\right)\|y\|. \end{aligned}$$

Thus for all $x \in [a, \eta]$ we always have

$$y(x) \geq \min\left\{\frac{x-a}{\eta-a}, 1 - \frac{x-a}{\eta-a}\right\}\|y\|.$$

Therefore we get

$$\min_{x\in[a+\frac{\beta-\alpha}{1-\alpha}(\eta-a),\tau]} y(x) \geq \min\left\{\frac{\beta-\alpha}{1-\alpha}, \frac{\eta-\tau}{\eta-a}\right\}\|y\| = \mu\|y\|.$$

This completes the proof. □

Now, we obtain the inequality for the Green's function of the problem (2.1).

Lemma 2.4. *Let $s_1 \in [\tau, \eta]$ and $s_2 \in [\eta, b]$. Then*

$$G(x, s_1) \geq \Lambda G(x, s_2), \quad x \in [a, b]. \tag{2.3}$$

Proof. Step 1. If $x \leq \eta$, then we have

$$\frac{G(x,s_1)}{G(x,s_2)} = \frac{G_1(x,s_1)}{G_1(x,s_2)},$$

$$G_1(x,s_1) = \begin{cases} g_{11}(x,s_1), & a \leq x \leq \eta, \\ g_{12}(x,s_1), & \eta < x \leq b, \end{cases} \quad \text{and} \quad G_1(x,s_2) = g_{13}(x,s_2).$$

Since $s_1 \in [\tau, \eta]$, $s_2 \in [\eta, b]$, we obtain

$$\begin{aligned} g_{11}(x,s_1) &= \frac{[\beta(x-\eta) + b - x](s_1 - a)}{\alpha(\eta - b) + \beta(a-\eta) + b - a} \\ &\geq \frac{[\beta(x-\eta) + b - x](\tau - a)}{\alpha(\eta - b) + \beta(a-\eta) + b - a} \\ &\geq \frac{(b-\eta)(\tau - a)}{\alpha(\eta - b) + \beta(a-\eta) + b - a}, \end{aligned} \tag{2.4}$$

$$
\begin{aligned}
g_{13}(x,s_2) &= \frac{[\alpha(\eta-x)+x-a](b-s_2)}{\alpha(\eta-b)+\beta(a-\eta)+b-a} \\
&\le \frac{[\alpha(\eta-x)+x-a](b-\eta)}{\alpha(\eta-b)+\beta(a-\eta)+b-a} \\
&\le \frac{(\eta-a)(b-\eta)}{\alpha(\eta-b)+\beta(a-\eta)+b-a}.
\end{aligned} \tag{2.5}
$$

Again,

$$
g_{12}(x,s_1) = \frac{\alpha(b-\eta)(s_1-x)+[\beta(s_1-\eta)+b-s_1](x-a)}{\alpha(\eta-b)+\beta(a-\eta)+b-a}.
$$

Thus, we consider two subcases.

(i) If $a+\frac{\alpha}{1-\beta}(\sigma(b)-\eta)\le\eta$, then for all $a\le x\le a+\frac{\alpha}{1-\beta}(\sigma(b)-\eta)$ we have

$$
\begin{aligned}
g_{12}(x,s_1) &\ge \frac{\alpha(b-\eta)(\tau-x)+[\beta(\tau-\eta)+b-\tau](x-a)}{\alpha(\eta-b)+\beta(a-\eta)+b-a} \\
&\ge \frac{\alpha(b-\eta)(\tau-a)}{\alpha(\eta-b)+\beta(a-\eta)+b-a}
\end{aligned}
$$

and for all $a+\frac{\alpha}{1-\beta}(\sigma(b)-\eta)\le x\le\eta$, we have

$$
\begin{aligned}
g_{12}(x,s_1) &\ge \frac{(b-\eta)[\alpha(\eta-x)+\beta(x-a)]}{\alpha(\eta-b)+\beta(a-\eta)+b-a} \\
&\ge \frac{\alpha(b-\eta)(\eta-a)}{\alpha(\eta-b)+\beta(a-\eta)+b-a} \\
&\ge \frac{\alpha(b-\eta)(\tau-a)}{\alpha(\eta-b)+\beta(a-\eta)+b-a}.
\end{aligned}
$$

(ii) If $\eta\le a+\frac{\alpha}{1-\beta}(\sigma(b)-\eta)$, then for all $a\le x\le\eta$, we have

$$
\begin{aligned}
g_{12}(x,s_1) &\ge \frac{\alpha(b-\eta)(\tau-x)+[\beta(\tau-\eta)+b-\tau](x-a)}{\alpha(\eta-b)+\beta(a-\eta)+b-a} \\
&\ge \frac{\alpha(b-\eta)(\tau-a)}{\alpha(\eta-b)+\beta(a-\eta)+b-a}.
\end{aligned}
$$

So, in either subcase, for all $s_1\in[a,\eta]$ we always have

$$
g_{12}(x,s_1) \ge \frac{\alpha(b-\eta)(\tau-a)}{\alpha(\eta-b)+\beta(a-\eta)+b-a}. \tag{2.6}
$$

Therefore, from (2.4)–(2.6) we get

$$
\begin{aligned}
&\frac{g_{12}(x,s_1)}{g_{13}(x,s_2)} \ge \frac{\alpha(b-\eta)(\tau-a)}{(\eta-a)(b-\eta)} \ge \alpha\frac{\tau-a}{\eta-a} \ge \alpha\frac{\beta-\alpha}{1-\alpha} \ge \Lambda, \\
&\frac{g_{11}(x,s_1)}{g_{13}(x,s_2)} \ge \frac{\tau-a}{\eta-a} \ge \frac{\beta-\alpha}{1-\alpha} \ge \Lambda,
\end{aligned}
$$

which yields

$$
\frac{G_1(x,s_1)}{G_1(x,s_2)} \ge \Lambda.
$$

Step 2. If $x \geq \eta$, then we have

$$\frac{G(x,s_1)}{G(x,s_2)} = \frac{G_2(x,s_1)}{G_2(x,s_2)},$$

$$G_2(x,s_1) = g_{21}(x,s_1) \quad \text{and} \quad G_2(x,s_2) = \begin{cases} g_{22}(x,s_2), \\ g_{23}(x,s_2). \end{cases}$$

Since $s_1 \in [\tau,\eta]$, $s_2 \in [\eta,b]$, we obtain

$$\begin{aligned} g_{21}(x,s_1) &= \frac{[\beta(x-\eta)+b-x](s_1-a)}{\alpha(\eta-b)+\beta(a-\eta)+b-a} \\ &\geq \frac{[\beta(x-\eta)+b-x](\tau-a)}{\alpha(\eta-b)+\beta(a-\eta)+b-a} \\ &\geq \frac{\beta(b-\eta)(\tau-a)}{\alpha(\eta-b)+\beta(a-\eta)+b-a}, \end{aligned} \tag{2.7}$$

$$\begin{aligned} g_{23}(x,s_2) &= \frac{[\alpha(\eta-x)+x-a](b-s_2)}{\alpha(\eta-b)+\beta(a-\eta)+b-a} \\ &\leq \frac{[\alpha(\eta-x)+x-a](b-\eta)}{\alpha(\eta-b)+\beta(a-\eta)+b-a} \\ &\leq \frac{[\alpha(\eta-b)+b-a](b-\eta)}{\alpha(\eta-b)+\beta(a-\eta)+b-a}. \end{aligned} \tag{2.8}$$

Again,

$$g_{22}(x,s_2) = \frac{(b-x)[\alpha(\eta-s_2)+s_2-a]+\beta(x-s_2)(\eta-a)}{\alpha(\eta-b)+\beta(a-\eta)+b-a}.$$

So, we consider two subcases.

(i) If $\eta \leq b - \frac{\beta}{1-\alpha}(\eta-a)$, then for all $\eta \leq x \leq b - \frac{\beta}{1-\alpha}(\eta-a)$, we have

$$g_{22}(x,s_2) \leq b-x \leq b-\eta,$$

and for all $b - \frac{\beta}{1-\alpha}(\eta-a) \leq x \leq b$, we have

$$g_{22}(x,s_2) \leq \frac{[b-x+\beta(x-\eta)](\eta-a)}{\alpha(\eta-b)+\beta(a-\eta)+b-a} \leq \frac{\beta(\eta-a)}{1-\alpha}.$$

(ii) If $b - \frac{\beta}{1-\alpha}(\eta-a) \leq \eta$, then for all $\eta \leq x \leq b$, we have

$$g_{22}(x,s_2) \leq \frac{[b-x+\beta(x-\eta)](\eta-a)}{\alpha(\eta-b)+\beta(a-\eta)+b-a} \leq \frac{\beta(\eta-a)}{1-\alpha}.$$

Thus, in either subcase, for all $s_2 \in [\eta,b]$ we always have

$$g_{22}(x,s_2) \leq \max\left\{b-\eta, \frac{\beta}{1-\alpha}(\eta-a)\right\}. \tag{2.9}$$

Therefore, from (2.7)–(2.9) we get

$$\begin{aligned} \frac{g_{21}(x,s_1)}{g_{22}(x,s_2)} &\geq \frac{\beta(b-\eta)(\tau-a)}{[\alpha(\eta-b)+\beta(a-\eta)+b-a]\max\left\{b-\eta, \frac{\beta}{1-\alpha}(\eta-a)\right\}} \\ &\geq \frac{\beta(b-\eta)(\beta-\alpha)(\eta-a)}{(1-\alpha)[\alpha(\eta-b)+\beta(a-\eta)+b-a]\max\left\{b-\eta, \frac{\beta}{1-\alpha}(\eta-a)\right\}} \\ &\geq \frac{\beta-\alpha}{b-a}\min\left\{\frac{\beta}{1-\alpha}(\eta-a), b-\eta\right\} \\ &\geq \Lambda, \end{aligned}$$

$$\frac{g_{21}(x,s_1)}{g_{23}(x,s_2)} \geq \frac{\beta(\tau-a)}{\alpha(\eta-b)+b-a} \geq \frac{\beta(\tau-a)}{b-a} \geq \frac{\beta}{b-a}.\frac{\beta-\alpha}{1-\alpha}(\eta-a) \geq \Lambda,$$

which yields

$$\frac{G_2(x,s_1)}{G_2(x,s_2)} \geq \Lambda.$$

Hence, (2.1) holds. $\square$

From Lemma 2.4, we get the following lemma.

Lemma 2.5. *Let conditions* (H1), (H2) *and* (H3) *hold. Then for all* $\Theta \in [0,\infty)$*,*

$$\int_\tau^\eta G(x,s)h^+(s)f(\Theta\gamma(s))\mathrm{d}s \geq \int_\eta^b G(x,s)h^-(s)f(\Theta\gamma(s))\mathrm{d}s.$$

Proof. For each $z \in [0, b-\eta]$, it can be seen that

$$\gamma\left(\eta - \frac{\eta-\tau}{b-\eta}z\right) = 1 - \frac{z}{b-\eta}(1-\alpha)\left(1 - \frac{\tau-a}{\eta-a}\right),$$
$$\gamma(\eta+z) = 1 - \frac{z}{b-\eta}(1-\beta).$$

From the fact that $\tau \in (a + \frac{\beta-\alpha}{1-\alpha}(\eta-a), \eta)$ and f is nondecreasing, for each $z \in [0, b-\eta]$, we get

$$f\left[1 - \frac{z}{b-\eta}(1-\alpha)\left(1-\frac{\tau-a}{\eta-a}\right)\right] \geq f\left[1 - \frac{z}{b-\eta}(1-\beta)\right].$$

Now, set $s = \eta - \delta z,\ z \in [0, b-\eta]$. For all $\Theta \in [0,\infty)$, from Lemma 2.4 and condition (H3), we have

$$\begin{aligned}
\int_\tau^\eta G(x,s)h^+(s)f(\Theta\gamma(s))\mathrm{d}s &= -\delta\int_{b-\eta}^0 G(x,\eta-\delta z)h^+(\eta-\delta z)f(\Theta\gamma(\eta-\delta z))\mathrm{d}z\\
&= \delta\int_0^{b-\eta} G(x,\eta-\delta z)h^+(\eta-\delta z)f\left[\left(1-\frac{z}{b-\eta}(1-\alpha)\left(1-\frac{\tau-a}{\eta-a}\right)\right)\Theta\right]\mathrm{d}z\\
&\geq \delta\Lambda\int_0^{b-\eta} G(x,\eta+z)h^+(\eta-\delta z)f\left[\left(1-\frac{z}{b-\eta}(1-\alpha)\left(1-\frac{\tau-a}{\eta-a}\right)\right)\Theta\right]\mathrm{d}z\\
&\geq \int_0^{b-\eta} G(x,\eta+z)h^-(\eta+z)f\left[\left(1-\frac{z}{b-\eta}(1-\beta)\right)\Theta\right]\mathrm{d}z.
\end{aligned}$$

Again, setting $s = \eta + z, z \in [0, b-\eta]$, for $\Theta \in [0,\infty)$, we obtain

$$\int_\eta^b G(x,s)h^-(s)f(\Theta\gamma(s))\mathrm{d}s = \int_0^{b-\eta} G(x,\eta+z)h^-(\eta+z)f\left[\left(1-\frac{z}{b-\eta}(1-\beta)\right)\Theta\right]\mathrm{d}z.$$

This completes the proof. $\square$

Now we are ready to prove that the operator A is completely continuous.

Lemma 2.6. *Assume that conditions* (H1), (H2) *and* (H3) *are satisfied. Then the operator* A *is completely continuous.*

Proof. At first, we show that $A : \mathcal{P} \to \mathcal{P}$. For all $y \in \mathcal{P}$, from Lemmas 2.2 and 2.5, and the fact that f is nondecreasing, we have

$$\begin{aligned}
\int_\tau^b G(x,s)h(s)f(y(s))\mathrm{d}s &= \int_\tau^\eta G(x,s)h^+(s)f(y(s))\mathrm{d}s - \int_\eta^b G(x,s)h^-(s)f(y(s))\mathrm{d}s\\
&\geq \int_\tau^\eta G(x,s)h^+(s)f(\gamma(s)y(\eta))\mathrm{d}s - \int_\eta^b G(x,s)h^-(s)f(\gamma(s)y(\eta))\mathrm{d}s\\
&\geq 0,
\end{aligned}$$

and thus

$$\begin{aligned}(Ay)(x) &= \int_a^b G(x,s)h(s)f(y(s))\mathrm{d}s \\ &= \int_a^\tau G(x,s)h^+(s)f(y(s))\mathrm{d}s + \int_\tau^b G(x,s)h(s)f(y(s))\mathrm{d}s \\ &\geq \int_a^\tau G(x,s)h^+(s)f(y(s))\mathrm{d}s \geq 0.\end{aligned}$$

Moreover, in view of $(Ay)(a) = \alpha(Ay)(\eta)$, $(Ay)(b) = \beta(Ay)(\eta)$, it follows that $A : \mathcal{P} \to \mathcal{C}_0^+[a,b]$. On the other hand,

$$\begin{aligned}&(Ay)''(x) = -h^+(x)f(y(x)) \leq 0, \quad x \in [a,\eta], \\ &(Ay)''(x) = h^-(x)f(y(x)) \geq 0, \quad x \in [\eta,b].\end{aligned}$$

This shows that $A : \mathcal{P} \to \mathcal{P}$.

It can be shown that $A : \mathcal{P} \to \mathcal{P}$ is completely continuous by Arzela–Ascoli theorem.

This completes the proof. □

3. Existence of one positive solution

Define the nonnegative extended real numbers, f_0 and f_∞, by

$$f_0 := \lim_{x\to 0^+} \frac{f(x)}{x}, \qquad f_\infty := \lim_{x\to\infty} \frac{f(x)}{x}.$$

Set

$$\begin{aligned}\Lambda_1 &= \mu \max_{x\in[a,b]} \int_{a+\frac{\beta-\alpha}{1-\alpha}(\eta-a)}^{\tau} G(x,s)h^+(s)\mathrm{d}s, \\ \Lambda_2 &= \max_{x\in[a,b]} \int_a^\eta G(x,s)h^+(s)\mathrm{d}s.\end{aligned}$$

Liu [6] has shown the existence of at least one positive solution of the BVP (1.1′) for λ belonging to certain intervals. In this paper, we investigate the existence of at least one positive solution for the BVP (1.1) when f is superlinear ($f_0 = 0$, $f_\infty = \infty$) or sublinear ($f_0 = \infty$, $f_\infty = 0$).

To prove the existence of at least one positive solution to the boundary value problem (1.1) as in BVP (1.1′), we need the following fixed-point theorem.

Theorem 3.1 ([7]). *Let E be a Banach space, and let $K \subset E$ be a cone. Assume Ω_1 and Ω_2 are open bounded subsets of E with $0 \in \Omega_1$, $\overline{\Omega}_1 \subset \Omega_2$, and let*

$$A : K \cap (\overline{\Omega}_2 \setminus \Omega_1) \to K$$

be a completely continuous operator such that either

(i) $\|Au\| \leq \|u\|$, $u \in K \cap \partial\Omega_1$, $\|Au\| \geq \|u\|$, $u \in K \cap \partial\Omega_2$; *or*
(ii) $\|Au\| \geq \|u\|$, $u \in K \cap \partial\Omega_1$, $\|Au\| \leq \|u\|$, $u \in K \cap \partial\Omega_2$,

hold. Then A has a fixed point in $K \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

Theorem 3.2. *Assume that conditions* (H1)–(H3) *are satisfied. If either*

(i) $f_0 = 0$ *and* $f_\infty = \infty$ *(f is superlinear), or*
(ii) $f_0 = \infty$ *and* $f_\infty = 0$ *(f is sublinear),*

then the second-order boundary value problem (1.1) *has at least one positive solution.*

Proof. First suppose f is superlinear. Since $f_0 = 0$, we may choose $H_1 > 0$ so that $f(y) \leq \epsilon y$ for $0 < y \leq H_1$, where $\epsilon > 0$ satisfies

$$\epsilon \Lambda_2 \leq 1.$$

If $y \in \mathcal{P}$ with $\|y\| = H_1$, then

$$\begin{aligned}(Ay)(x) &= \int_a^b G(x,s)h(s)f(y(s))\mathrm{d}s \\ &= \int_a^\eta G(x,s)h^+(s)f(y(s))\mathrm{d}s - \int_\eta^b G(x,s)h^-(s)f(y(s))\mathrm{d}s \\ &\leq \int_a^\eta G(x,s)h^+(s)f(y(s))\mathrm{d}s \\ &\leq \epsilon\|y\| \int_a^\eta G(x,s)h^+(s)\mathrm{d}s \\ &\leq \epsilon\|y\|\Lambda_2.\end{aligned}$$

Consequently, $\|Ay\| \leq \|y\|$. So, if we set

$$\Omega_1 := \{y \in \mathcal{P} : \|y\| < H_1\},$$

then $\|Ay\| \leq \|y\|$ for $y \in \mathcal{P} \cap \partial\Omega_1$. Further, since $f_\infty = \infty$, there exists $\widehat{H}_2 > 0$ such that $f(y) \geq \rho y$, for $y \geq \widehat{H}_2$, where $\rho > 0$ is chosen so that

$$\rho \Lambda_1 \geq 1.$$

Let $H_2 = \max\{2H_1, \frac{\widehat{H}_2}{\mu}\}$ and set

$$\Omega_2 := \{y \in \mathcal{P} : \|y\| < H_2\}.$$

If $y \in \mathcal{P}$ with $\|y\| = H_2$, then

$$y(t) \geq \mu\|y\| = \mu H_2 \geq \widehat{H}_2, \quad x \in \left[a + \frac{\beta - \alpha}{1 - \alpha}(\eta - a), \tau\right]$$

and so

$$\begin{aligned}\|Ay\| &= \max_{x\in[a,b]} \left[\int_a^\tau G(x,s)h^+(s)f(y(s))\mathrm{d}s + \int_\tau^b G(x,s)h(s)f(y(s))\mathrm{d}s\right] \\ &\geq \max_{x\in[a,b]} \int_a^\tau G(x,s)h^+(s)f(y(s))\mathrm{d}s \\ &\geq \max_{x\in[a,b]} \int_{a+\frac{\beta-\alpha}{1-\alpha}(\eta-a)}^\tau G(x,s)h^+(s)f(y(s))\mathrm{d}s \\ &\geq \rho\|y\|\mu \max_{x\in[a,b]} \int_{a+\frac{\beta-\alpha}{1-\alpha}(\eta-a)}^\tau G(x,s)h^+(s)\mathrm{d}s \\ &\geq \rho\Lambda_1\|y\| \geq \|y\|.\end{aligned}$$

Hence $\|Ay\| \geq \|y\|$, $y \in \mathcal{P} \cap \partial\Omega_2$. Thus by the first part of Theorem 3.1, A has a fixed point y in $\mathcal{P} \cap (\overline{\Omega}_2 \setminus \Omega_1)$ with $H_1 \leq \|y\| \leq H_2$.

Now suppose f is sublinear. Since $f_0 = \infty$, we choose $H_3 > 0$ such that $f(y) \geq Ly$, for $0 < y \leq H_3$, where $L > 0$ satisfies

$$L\Lambda_1 \geq 1.$$

If $y \in \mathcal{P}$ with $\|y\| = H_3$, then

$$\|Ay\| \geq \max_{x\in[a,b]} \int_{a+\frac{\beta-\alpha}{1-\alpha}(\eta-a)}^{\tau} G(x,s)h^+(s)f(y(s))\mathrm{d}s$$
$$\geq L\Lambda_1\|y\| \geq \|y\|.$$

Hence $\|Ay\| \geq \|y\|$. So, if we set

$$\Omega_3 := \{y \in \mathcal{P} : \|y\| < H_3\},$$

then $\|Ay\| \geq \|y\|$ for $y \in \mathcal{P} \cap \partial\Omega_3$. Now, since $f_\infty = 0$, there exists $\widehat{H}_4 > 0$ such that $f(y) \leq \lambda y$, for $y \geq \widehat{H}_4$, where $\lambda > 0$ is chosen so that

$$\lambda\Lambda_2 \leq 1.$$

We consider two subcases. The first case is that f is bounded. In this case there is a positive number N such that $f(y) \leq N$ for $y \in [0,\infty)$. Let $H_4 = \max\{2H_3, N\Lambda_2\}$ and set

$$\Omega_4 := \{y \in \mathcal{P} : \|y\| < H_4\}.$$

Then, for $y \in \mathcal{P}$, with $\|y\| = H_4$, we have

$$(Ay)(x) \leq \int_a^\eta G(x,s)h^+(s)f(y(s))\mathrm{d}s$$
$$\leq N \max_{x\in[a,b]} \int_a^\eta G(x,s)h^+(s)\mathrm{d}s$$
$$\leq N\Lambda_2 \leq H_4 = \|y\|.$$

It follows that if $y \in \mathcal{P} \cap \partial\Omega_4$, then $\|Ay\| \leq \|y\|$.

Next we consider the case where f is unbounded. Let $H_4 = \max\{2H_3, \widehat{H}_4\}$ be such that $f(y) \leq f(H_4)$ for $0 \leq y \leq H_4$. For $y \in \mathcal{P}$, with $\|y\| = H_4$,

$$(Ay)(x) \leq \int_a^\eta G(x,s)h^+(s)f(y(s))\mathrm{d}s$$
$$\leq f(H_4) \max_{x\in[a,b]} \int_a^\eta G(x,s)h^+(s)\mathrm{d}s$$
$$\leq \lambda H_4\Lambda_2 \leq \|y\|$$

so that $\|Ay\| \leq \|y\|$. For this case, if we let

$$\Omega_4 := \{y \in \mathcal{P} : \|y\| < H_4\},$$

then $\|Ay\| \leq \|y\|$, for $y \in \mathcal{P} \cap \partial\Omega_4$.

By the second part of Theorem 3.1, A has a fixed point y in $\mathcal{P} \cap (\overline{\Omega}_4 \setminus \Omega_3)$ such that $H_3 \leq \|y\| \leq H_4$. This completes the sublinear part of the theorem. Then, the boundary value problem (1.1) has at least one positive solution. □

4. Existence of two positive solutions

In this section, using Theorem 4.1 (Avery–Henderson fixed-point theorem) we prove the existence of at least two positive solutions of the boundary value problem (1.1) which has not been studied by Liu [6].

Theorem 4.1 ([8]). *Let $\mathcal{P}$ be a cone in a real Banach space S. If η and ψ are increasing, nonnegative continuous functionals on $\mathcal{P}$, let θ be a nonnegative continuous functional on $\mathcal{P}$ with $\theta(0) = 0$ such that, for some positive constants r and M,*

$$\psi(u) \leq \theta(u) \leq \eta(u) \quad \textit{and} \quad \|u\| \leq M\psi(u)$$

for all $u \in \overline{\mathcal{P}(\psi, r)}$. *Suppose that there exist positive numbers* $p < q < r$ *such that*

$$\theta(\lambda u) \leq \lambda\theta(u), \quad \text{for all } 0 \leq \lambda \leq 1 \text{ and } u \in \partial P(\theta, q).$$

If $A : \overline{\mathcal{P}(\psi, r)} \to \mathcal{P}$ *is a completely continuous operator satisfying*

(i) $\psi(Au) > r$ *for all* $u \in \partial\mathcal{P}(\psi, r)$,
(ii) $\theta(Au) < q$ *for all* $u \in \partial\mathcal{P}(\theta, q)$,
(iii) $\mathcal{P}(\eta, p) \neq \{\}$ *and* $\eta(Au) > p$ *for all* $u \in \partial\mathcal{P}(\eta, p)$,

then A has at least two fixed points u_1 *and* u_2 *such that*

$$p < \eta(u_1) \quad \text{with } \theta(u_1) < q \quad \text{and} \quad q < \theta(u_2) \quad \text{with } \psi(u_2) < r.$$

Define constants

$$m := \left(\max_{x\in\left[a+\frac{\beta-\alpha}{1-\alpha}(\eta-a),\tau\right]} \int_a^\eta G(x,s)h^+(s)\mathrm{d}s \right)^{-1}, \tag{4.1}$$

$$M := \mu \int_{a+\frac{\beta-\alpha}{1-\alpha}(\eta-a)}^{\tau} G\left(a + \frac{\beta-\alpha}{1-\alpha}(\eta-a), s\right) h^+(s)\mathrm{d}s. \tag{4.2}$$

Theorem 4.2. *Assume* (H1)–(H3) *hold. Suppose there exist positive numbers* $0 < p < q < r$ *such that the function f satisfies the following conditions:*

(i) $f(y) > p/M$ *for* $y \in [\mu p, p]$,
(ii) $f(y) < qm$ *for* $y \in [0, q/\mu]$,
(iii) $f(y) > r/M$ *for* $y \in [r, r/\mu]$,

where μ, m, M *are as defined in* (2.2), (4.1) *and* (4.2) *respectively. Then the boundary value problem* (1.1) *has at least two positive solutions* y_1 *and* y_2 *such that*

$$p < \max_{x\in[a,b]} y_1(x) \quad \text{with} \max_{x\in\left[a+\frac{\beta-\alpha}{1-\alpha}(\eta-a),\tau\right]} y_1(x) < q,$$

$$q < \max_{x\in\left[a+\frac{\beta-\alpha}{1-\alpha}(\eta-a),\tau\right]} y_2(x) \quad \text{with} \min_{x\in\left[a+\frac{\beta-\alpha}{1-\alpha}(\eta-a),\tau\right]} y_2(x) < r.$$

Proof. Let the nonnegative, increasing, continuous functionals ψ, θ, and η be defined on the cone $\mathcal{P}$ by

$$\psi(y) := \min_{x\in\left[a+\frac{\beta-\alpha}{1-\alpha}(\eta-a),\tau\right]} y(x), \qquad \theta(y) := \max_{x\in\left[a+\frac{\beta-\alpha}{1-\alpha}(\eta-a),\tau\right]} y(x), \qquad \eta(y) := \max_{x\in[a,b]} y(x)$$

and let $\mathcal{P}(\psi, r) := \{y \in \mathcal{P} : \psi(y) < r\}$.

For each $y \in \mathcal{P}$ we have

$$\psi(y) \leq \theta(y) \leq \eta(y), \tag{4.3}$$

$$\|y\| \leq \frac{1}{\mu} \min_{x\in\left[a+\frac{\beta-\alpha}{1-\alpha}(\eta-a),\tau\right]} y(x) = \frac{1}{\mu}\psi(y) \leq \frac{1}{\mu}\theta(y) \leq \frac{1}{\mu}\eta(y). \tag{4.4}$$

For any $y \in \mathcal{P}$, (4.3) and (4.4) imply

$$\psi(y) \leq \theta(y) \leq \eta(y), \qquad \|y\| \leq \frac{1}{\mu}\psi(y).$$

For all $y \in \mathcal{P}$, $\lambda \in [0, 1]$ we have

$$\theta(\lambda y) = \max_{x\in\left[a+\frac{\beta-\alpha}{1-\alpha}(\eta-a),\tau\right]} (\lambda y)(x) = \lambda \max_{x\in\left[a+\frac{\beta-\alpha}{1-\alpha}(\eta-a),\tau\right]} y(x) = \lambda\theta(y).$$

It is clear that $\theta(0) = 0$.

We now show that the remaining conditions of Theorem 4.1 are satisfied.

Firstly, we shall verify that the condition (iii) of Theorem 4.1 is satisfied. Since $0 \in \mathcal{P}$ and $p > 0$, $\mathcal{P}(\eta, p) \neq \{\}$. Since $y \in \partial\mathcal{P}(\eta, p)$, $\mu p \leq y(x) \leq \|y\| = p$ for $x \in [a + \frac{\beta-\alpha}{1-\alpha}(\eta - a), \tau]$. Therefore,

$$\begin{aligned}
\eta(Ay) &= \max_{x\in[a,b]} Ay(x) \\
&> \mu Ay\left(a + \frac{\beta-\alpha}{1-\alpha}(\eta - a)\right) \\
&\geq \mu \int_{a+\frac{\beta-\alpha}{1-\alpha}(\eta-a)}^{\tau} G\left(a + \frac{\beta-\alpha}{1-\alpha}(\eta - a), s\right) h^{+}(s) f(y(s))\mathrm{d}s \\
&> \mu \frac{p}{M} \int_{a+\frac{\beta-\alpha}{1-\alpha}(\eta-a)}^{\tau} G\left(a + \frac{\beta-\alpha}{1-\alpha}(\eta - a), s\right) h^{+}(s)\mathrm{d}s = p,
\end{aligned}$$

using (4.2) and hypothesis (i).

Now we shall show that the condition (ii) of Theorem 4.1 is satisfied. Since $y \in \partial\mathcal{P}(\theta, q)$, from (4.4) we have that $0 \leq y(x) \leq \|y\| \leq q/\mu$ for $x \in [a, b]$. Thus

$$\begin{aligned}
\theta(Ay) &= \max_{x\in\left[a+\frac{\beta-\alpha}{1-\alpha}(\eta-a),\tau\right]} Ay(x) \\
&= \max_{x\in\left[a+\frac{\beta-\alpha}{1-\alpha}(\eta-a),\tau\right]} \int_a^b G(x, s)h(s) f(y(s))\mathrm{d}s \\
&\leq qm \max_{x\in\left[a+\frac{\beta-\alpha}{1-\alpha}(\eta-a),\tau\right]} \int_a^b G(x, s)h(s)\mathrm{d}s \\
&< qm \max_{x\in\left[a+\frac{\beta-\alpha}{1-\alpha}(\eta-a),\tau\right]} \int_a^{\eta} G(x, s)h^{+}(s)\mathrm{d}s = q,
\end{aligned}$$

by hypothesis (ii) and (4.1).

Lastly using hypothesis (iii) and (4.3), we shall show that the condition (i) of Theorem 4.1 is satisfied. Since $y \in \partial\mathcal{P}(\psi, r)$, from (4.4) we have that $\min_{x\in[a+\frac{\beta-\alpha}{1-\alpha}(\eta-a),\tau]} y(x) = r$ and $r \leq \|y\| \leq r/\mu$. By concavity of Ay,

$$\begin{aligned}
\psi(Ay) &= \min_{x\in\left[a+\frac{\beta-\alpha}{1-\alpha}(\eta-a),\tau\right]} Ay(x) \\
&\geq \mu\|Ay\| \\
&\geq \mu Ay\left(a + \frac{\beta-\alpha}{1-\alpha}(\eta - a)\right) \\
&= \mu \int_a^b G\left(a + \frac{\beta-\alpha}{1-\alpha}(\eta - a), s\right) h(s) f(y(s))\mathrm{d}s \\
&\geq \mu \int_{a+\frac{\beta-\alpha}{1-\alpha}(\eta-a)}^{\tau} G\left(a + \frac{\beta-\alpha}{1-\alpha}(\eta - a), s\right) h^{+}(s) f(y(s))\mathrm{d}s \\
&> \frac{r}{M}\mu \int_{a+\frac{\beta-\alpha}{1-\alpha}(\eta-a)}^{\tau} G\left(a + \frac{\beta-\alpha}{1-\alpha}(\eta - a), s\right) h^{+}(s)\mathrm{d}s = r,
\end{aligned}$$

using (4.2) and hypothesis (iii). This completes the proof. □

5. Examples

In the case where h is smooth, the condition (H3) is not satisfied. Therefore we cannot make any conclusion about the existence of positive solutions for BVP (1.1).

Example 5.1. Consider the boundary value problem

$$\begin{cases} y''(x) + h(x)(1+\sqrt{y}) = 0, & x \in [0,2], \\ y(0) = \dfrac{1}{10} y(1), & y(2) = \dfrac{1}{5} y(1), \end{cases} \tag{5.1}$$

where $h(x) = \begin{cases} 245(1-x), 0 \le x \le 1, \\ 2(1-x)^3, 1 \le x \le 2. \end{cases}$

Then we have $a = 0, \eta = 1, b = 2, \alpha = \frac{1}{10}, \beta = \frac{1}{5}, \delta = \frac{6}{7},\ \tau = \frac{1}{7},\ \Lambda = \frac{1}{90},\ f(y) = 1+\sqrt{y},\ y \in [0,\infty)$; and $H(x) = 180x(1-x^2) \ge 0, x \in [0,1]$.

It yields $f_0 = +\infty$, and $f_\infty = 0$.

Thus the boundary value problem (5.1) has at least one positive solution by Theorem 3.2.

Example 5.2. Let us introduce an example to illustrate the usage of Theorem 4.2.

Consider the boundary value problem

$$\begin{cases} y''(x) + h(x)\dfrac{10(y+1)^3}{8((y+1)^2+996)} = 0, & x \in \left[1, \dfrac{5}{2}\right], \\ y(1) = \dfrac{1}{3} y\left(\dfrac{7}{4}\right), & y\left(\dfrac{5}{2}\right) = \dfrac{5}{9} y\left(\dfrac{7}{4}\right), \end{cases} \tag{5.2}$$

where $h(x) = \begin{cases} \frac{1989}{20}(7-4x), 1 \le x \le \frac{7}{4}, \\ 7-4x, \frac{7}{4} \le x \le \frac{5}{2}. \end{cases}$

Then we have $a = 1, \eta = \frac{7}{4}, b = \frac{5}{2}, \alpha = \frac{1}{3}, \beta = \frac{5}{9}, \tau = \frac{3}{2}, \delta = \frac{1}{3}, \Lambda = \frac{5}{54}, f(y) = \frac{10(y+1)^3}{8((y+1)^2+996)}, y \ge 0$; and $H(x) = x \ge 0, x \in [0, \frac{7}{4}]$.

Clearly f is continuous and increasing on $[0, \infty)$.

By (2.2), (4.1) and (4.2), we get $\mu = \frac{1}{3}, m = 0.0358, M = 3.8559$. If we take $p = \frac{1}{10^3}, q = \frac{1}{3}$, and $r = 19$, then

$$0 < p < q < r.$$

It is clear that (i), (ii), (iii) of Theorem 4.2 are satisfied. So the boundary value problem (5.2) has at least two positive solutions y_1, y_2 satisfying

$$\frac{1}{10^3} < \max_{x \in [1, \frac{5}{2}]} y_1(x) \quad \text{with} \quad \max_{x \in [\frac{5}{4}, \frac{3}{2}]} y_1(x) < \frac{1}{3},$$
$$\frac{1}{3} < \max_{x \in [\frac{5}{4}, \frac{3}{2}]} y_2(x) \quad \text{with} \quad \min_{x \in [\frac{5}{4}, \frac{3}{2}]} y_2(x) < 19.$$

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